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Vector Variational Inequalities for Nondifferentiable Convex Vector Optimization Problems★

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Abstract. In this paper, we consider a nondifferentiable convex vector optimization problem (VP), and formulate several kinds of vector variational inequalities with subdifferentials. Here we examine relations among solution sets of such vector variational inequalities and (VP).

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1. Introduction and preliminary results

We consider the following scalar convex optimization problem.

(SP) Minimize $f(x)$
subject to $x \in D$,

where $f: \mathbb{R}^n \to \mathbb{R}$ is a convex function and D is a convex subset of $\mathbb{R}^n$. The subdifferential of f at $x \in \mathbb{R}^n$ is defined as follows: $\partial f(x) = \{\xi \in \mathbb{R}^n \mid f(y) \geqslant f(x) + \langle \xi, y - x \rangle \quad \forall y \in \mathbb{R}^n\}$.

We can consider two variational inequalities for (SP)

(VI) Find $\bar{x} \in D$ such that $\exists \xi \in \partial f(\bar{x})$ such that $\langle \xi, x - \bar{x} \rangle \geqslant 0 \quad \forall x \in D$.

(MVI) Find $\bar{x} \in D$ such that $\forall x \in D$, $\forall \xi \in \partial f(x)$ $\langle \xi, x - \bar{x} \rangle \geqslant 0$.

We denote the solution sets of (SP), (VI) and (MVI) by sol(SP), sol(VI) and sol(MVI), respectively.

Then it is well known that

$$sol(\text{SP}) = sol(\text{VI}) = sol(\text{MVI}).$$

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This means that variational inequality can be a strong tool for studying the solution set of (SP).

Now we consider the following vector optimization problem

$$\text{(VP)} \quad \begin{array}{ll} \text{Minimize} & f(x) := (f_1(x), \ldots, f_p(x)) \\ \text{subject to} & x \in D, \end{array}$$

where $f_i: \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, p$, are functions and D is a subset of $\mathbb{R}^n$.

Solving (VP) means to find the (properly, weakly) efficient solutions which are defined as follows.

DEFINITION 1.1. (1) $\bar{x} \in D$ is said to be an efficient solution of (VP) if for any $x \in D$,

$$(f_1(x) - f_1(\bar{x}), \ldots, f_p(x) - f_p(\bar{x})) \notin -\mathbb{R}^p_+ \setminus \{0\},$$

where $\mathbb{R}^p_+$ is the nonnegative orthant of $\mathbb{R}^p$.

(2) $\bar{x} \in D$ is called a properly efficient solution of (VP) if $\bar{x} \in D$ is an efficient solution of (VP) and there exists a constant $M > 0$ such that for each $i = 1, \ldots, p$, we have

$$\frac{f_i(\bar{x}) - f_i(x)}{f_j(x) - f_j(\bar{x})} \leqslant M$$

for some j such that $f_j(x) > f_j(\bar{x})$ whenever $x \in D$ and $f_i(x) < f_i(\bar{x})$.

(3) $\bar{x} \in D$ is said to be a weakly efficient solution of (VP) if for any $x \in D$,

$$(f_1(x) - f_1(\bar{x}), \ldots, f_p(x) - f_p(\bar{x})) \notin -\text{int } \mathbb{R}^p_+,$$

where int $\mathbb{R}^p_+$ is the interior of $\mathbb{R}^p_+$.

We denote the set of all the efficient solution of (VP), the set of all the weakly efficient solution of (VP), the set of all the properly efficient solution of (VP) by Eff(VP), $WEff$(VP) and $PrEff$(VP), respectively.

It is clear that $PrEff(\text{VP}) \subset Eff(\text{VP}) \subset WEff(\text{VP})$. For basic meanings and properties of such solution sets, see [1].

Throughout this paper, we will assume that the objective functions $f_i, i = 1, \ldots, p$, are convex and the constraint set D is a closed convex subset of $\mathbb{R}^n$.

Recently, Giannessi [2] considered the following vector variational inequalities for a differentiable convex vector optimization (VP) (when f_i, $i = 1, \ldots, p$, are differentiable)

$$(\text{VVI})_\nabla \qquad \begin{array}{l} \text{Find } \bar{x} \in D \text{ such that} \\ (\langle \nabla f_1(\bar{x}), x - \bar{x} \rangle, \ldots, \langle \nabla f_p(\bar{x}), x - \bar{x} \rangle) \notin -\mathbb{R}^p_+ \setminus \{0\}, \quad \forall x \in D, \end{array}$$

where $\nabla f_i(x)$ is the gradient of f_i at x and $\langle\cdot,\cdot\rangle$ is the scalar product on $\mathbb{R}^n$.

$(\text{MVVI})_\nabla$ Find $\bar{x}\in D$ such that $(\langle\nabla f_1(x), x-\bar{x}\rangle,\dots,\langle\nabla f_p(x), x-\bar{x}\rangle)\notin -\mathbb{R}^p_+\backslash\{0\}, \forall x\in D.$

$(\text{WVVI})_\nabla$ Find $\bar{x}\in D$ such that $(\langle\nabla f_1(\bar{x}), x-\bar{x}\rangle,\dots,\langle\nabla f_p(\bar{x}), x-\bar{x}\rangle)\notin -\text{int }\mathbb{R}^p_+, \quad \forall x\in D.$ where int $\mathbb{R}^p_+$ is the interior of $\mathbb{R}^p_+$.

He proved that if f_i, $i=1,\dots,p$, are differentiable, then

$$sol(\text{VVI})_\nabla \subset sol(\text{MVVI})_\nabla = Eff(\text{VP}) \subset WEff(\text{VP}) = sol(\text{WVVI})_\nabla.$$

Being inspired by the above-mentioned Giannessi's result, many authors ([3–7]) have studied relations between vector variational inequalities and vector optimization problems.

In this paper, we consider scalar or vector variational inequalities for the nondifferentiable convex vector optimization problem (VP), which are formulated as below, and investigate relations among solution sets of such variational inequality problem and (VP). Our vector variational inequalities with subdifferentials can be regarded as special cases of usual ones with multifunctions. So, our results can be helpful for studying solution sets of nondifferentiable convex vector optimization problems and usual vector variational inequalities with multifunctions.

$(\text{VI})_\lambda$ Find $\bar{x}\in D$ such that $\exists\xi_i\in\partial f_i(\bar{x})$, $i=1,\dots,p$, such that $\langle\sum_{i=1}^p\lambda_i\xi_i, x-\bar{x}\rangle \geqslant 0 \quad \forall x\in D,$ where $\lambda=(\lambda_1,\dots,\lambda_p)\in\mathbb{R}^p_+\backslash\{0\}$.

$(\text{MVI})_\lambda$ Find $\bar{x}\in D$ such that $\forall x\in D$, $\exists\xi_i\in\partial f_i(x)$, $i=1,\dots,p$, $\langle\sum_{i=1}^p\lambda_i\xi_i, x-\bar{x}\rangle\geqslant 0.$

$(\text{VVI})_1$ Find $\bar{x}\in D$ such that $\forall x\in D$, $\forall\xi_i\in\partial f_i(\bar{x})$, $i=1,\dots,p$, $(\langle\xi_1, x-\bar{x}\rangle,\dots,\langle\xi_p, x-\bar{x}\rangle)\notin -\mathbb{R}^p_+\backslash\{0\}.$

$(\text{VVI})_2$ Find $\bar{x}\in D$ such that $\exists\xi_i\in\partial f_i(\bar{x})$, $i=1,\dots,p$, such that $(\langle\xi_1, x-\bar{x}\rangle,\dots,\langle\xi_p, x-\bar{x}\rangle)\notin -\mathbb{R}^p_+\backslash\{0\}, \quad \forall x\in D.$

$(\text{VVI})_3$ Find $\bar{x}\in D$ such that $\forall x\in D$, $\exists\xi_i\in\partial f_i(\bar{x})$, $i=1,\dots,p$, such that $(\langle\xi_1, x-\bar{x}\rangle,\dots,\langle\xi_p, x-\bar{x}\rangle)\notin -\mathbb{R}^p_+\backslash\{0\}.$

(MVVI) Find $\bar{x} \in D$ such that $\forall x \in D$, $\forall \xi_i \in \partial f_i(x)$, $i = 1, \ldots, p$, such that $(\langle \xi_1, x - \bar{x} \rangle, \ldots, \langle \xi_p, x - \bar{x} \rangle) \notin -\mathbb{R}^p_+ \setminus \{0\}$.

$(\mathrm{WVVI})_1$ Find $\bar{x} \in D$ such that $\forall x \in D$, $\forall \xi_i \in \partial f_i(\bar{x})$, $i = 1, \ldots, p$, $(\langle \xi_1, x - \bar{x} \rangle, \ldots, \langle \xi_p, x - \bar{x} \rangle) \notin -\mathrm{int}\ \mathbb{R}^p_+$.

$(\mathrm{WVVI})_2$ Find $\bar{x} \in D$ such that $\exists \xi_i \in \partial f_i(\bar{x})$, $i = 1, \ldots, p$, such that $(\langle \xi_1, x - \bar{x} \rangle, \ldots, \langle \xi_p, x - \bar{x} \rangle) \notin -\mathrm{int}\ \mathbb{R}^p_+ \quad \forall x \in D$.

$(\mathrm{WVVI})_3$ Find $\bar{x} \in D$ such that $\forall x \in D, \exists \xi_i \in \partial f_i(\bar{x})$, $i = 1, \ldots, p$, such that $(\langle \xi_1, x - \bar{x} \rangle, \ldots, \langle \xi_p, x - \bar{x} \rangle) \notin -\mathrm{int}\ \mathbb{R}^p_+$.

(WMVVI) Find $\bar{x} \in D$ such that $\forall x \in D$, $\forall \xi_i \in \partial f_i(x)$, $i = 1, \ldots, p$, such that $(\langle \xi_1, x - \bar{x} \rangle, \ldots, \langle \xi_p, x - \bar{x} \rangle) \notin -int\, \mathbb{R}^p_+$.

We denote the solution sets of the above inequality problems by $sol(\mathrm{VI})_\lambda$, $sol(\mathrm{MVI})_\lambda, sol(\mathrm{VVI}), \ldots, sol(\mathrm{WMVVI})$, respectively.

Now we give preliminary results which are needed in next sections.

LEMMA 1.1. [8] *$\bar{x} \in PrEff(VP)$ if and only if $\exists \lambda_i > 0, i = 1, \ldots, p$ such that $\bar{x}$ is a solution of the following scalar optimization problem*

$$\begin{array}{ll} \text{Minimize} & \sum_{i=1}^{p} \lambda_i f_i(x) \\ \text{subject to} & x \in D. \end{array}$$

LEMMA 1.2. [9] *If the objective functions $f_i, i = 1, \ldots, p$, are linear and the constraint set D is a polyhedral convex subset of $\mathbb{R}^n$, then $PrEff(VP) = Eff(VP)$.*

LEMMA 1.3. [10] *$\bar{x} \in WEff(VP)$ if and only if $\exists \lambda_i \geqslant 0$, $i = 1, \ldots, p$, $(\lambda_1, \ldots, \lambda_p) \neq 0$ such that $\bar{x}$ is a solution of the following scalar optimization problem*

$$\begin{array}{ll} \text{Minimize} & \sum_{i=1}^{p} \lambda_i f_i(x) \\ \text{subject to} & x \in D. \end{array}$$

LEMMA 1.4. *Let A be a convex subset of $\mathbb{R}^n$ and let B be a compact convex subset of $\mathbb{R}^n$. Assume that $0 \in A$. Then the following statements are equivalent*

(*i*) *$\exists b \in B$ such that $\langle b, a \rangle \geqslant 0 \quad \forall a \in A$.*
(*ii*) *$\forall a \in A$, $\exists b \in B$ such that $\langle b, a \rangle \geqslant 0$.*

Proof. Let $f(x)=\max_{b\in B}\langle b,x\rangle$. Then $f:\mathbb{R}^n\to\mathbb{R}$ is a convex function and $\partial f(0)=B$. Moreover, we have

$$\begin{aligned}
\text{(ii)} &\iff \max_{b\in B}\langle b,a\rangle\geqslant 0 \quad \forall a\in A,\\
&\iff f(a)\geqslant f(0) \quad \forall a\in A,\\
&\iff 0\in\partial f(0)+N_A(0), \text{where } N_A(0) \text{ is the normal cone to } A \text{ to } 0,\\
&\iff \exists b\in B \text{ such that } \langle b,a\rangle\geqslant 0 \quad \forall a\in A,\\
&\iff (i).
\end{aligned}$$ □

The following lemma is a generalized Gordan theorem for convex functions.

LEMMA 1.5. [11] *Let* $f_i:\mathbb{R}^n\to\mathbb{R},\ i=1,\dots,p$ *be convex functions and let* D *be a convex subset of* $\mathbb{R}^n$.

Then the following statements are equivalent

(*i*) *there is no* $x\in D$ *such that* $f_i(x)<0$ *for all* $i=1,\dots,p$.

(*ii*) $\exists\lambda_i\geqslant 0,\ i=1,\dots,p,\ (\lambda_1,\dots,\lambda_p)\neq 0$ *such that* $\sum_{i=1}^p\lambda_i f_i(x)\geqslant 0\quad\forall x\in D$.

2. Relations

Now we give relations among solution sets of the convex vector optimization problem (VP) and the vector variational inequality problems.

THEOREM 2.1. *The following are true*

(*1*) $sol(VVI)_1\subset sol(VVI)_2$.

(*2*) $PrEff(VP)=\bigcup_{\lambda\in\mathrm{int}\mathbb{R}^p_+} sol(VI)_\lambda\subset sol(VVI)_2\subset sol(VVI)_3$
$\subset sol(MVVI)=Eff(VP)$.

Proof. It is clear that $sol(\mathrm{VVI})_1\subset sol(\mathrm{VVI})_2$.

Now we prove that $PrEff(\mathrm{VP})=\bigcup_{\lambda\in\mathrm{int}\mathbb{R}^p_+} sol(\mathrm{VI})_\lambda$. By Lemma 1.1, $\bar{x}\in PrEff(\mathrm{VP})$ if and only if $\exists\lambda_i>0,\ i=1,\dots,p$, such that $\bar{x}\in D$ is a solution of the following scalar optimization problem (SP)

$$\text{(SP)}\quad \begin{array}{ll}\text{Minimize} & \sum_{i=1}^p\lambda_i f_i(x)\\ \text{subject to} & x\in D.\end{array}$$

Furthermore, it is well known that the fact that $\bar{x}\in D$ is a solution of (SP) is equivalent to $\bar{x}\in sol(\mathrm{VI})_\lambda$. We can easily check that

$$\bigcup_{\lambda\in\mathrm{int}\mathbb{R}^p_+} sol(\mathrm{VI})_\lambda\subset sol(\mathrm{VVI})_2\subset sol(\mathrm{VVI})_3.$$

From the monotonicity of the subdifferential of f_i, we can prove that $sol(\mathrm{VVI})_3 \subset sol(\mathrm{MVVI})$.

It was proved in Ref. [3] that $Eff(\mathrm{VP}) = sol(\mathrm{MVVI})$. For the completeness, we prove that $Eff(\mathrm{VP}) = sol(\mathrm{MVVI})$. It can be easily proved that $Eff(\mathrm{VP}) \subset sol(\mathrm{MVVI})$. Now we prove that $sol(\mathrm{MVVI}) \subset Eff(\mathrm{VP})$.

Let $\bar{x} \in sol(\mathrm{MVVI})$. Suppose to the contrary that $\bar{x} \notin Eff(\mathrm{VP})$. Then there exists $z \in D$ such that

$$(f_1(z) - f_1(\bar{x}), \ldots, f_p(z) - f_p(\bar{x})) \in -\mathbb{R}^p_+ \backslash \{0\}. \tag{2.1}$$

Since D is convex, we have $z(\alpha) := \alpha\bar{x} + (1-\alpha)z \in D$ for any $\alpha \in [0, 1]$. Since f_i is convex, $f_i(z(\alpha)) \leqslant \alpha f_i(\bar{x}) + (1-\alpha) f_i(z)$ for any $\alpha \in [0, 1]$ and hence $f_i(z(\alpha)) - f_i(\bar{x}) \leqslant (\alpha - 1)[f_i(\bar{x}) - f_i(z)]$ for any $\alpha \in [0, 1]$. So we have

$$\frac{f_i(z(\alpha)) - f_i(z(1))}{\alpha - 1} \geqslant f_i(\bar{x}) - f_i(z) \quad \text{for any } \alpha \in (0, 1).$$

By Lebourg's Mean Value Theorem in Ref. [12], there exist $\alpha_i \in (0, 1)$ and $\xi_i \in \partial f_i(z(\alpha_i))$, $i = 1, \ldots, p$, such that

$$\langle \xi_i, \bar{x} - z \rangle \geqslant f_i(\bar{x}) - f_i(z). \tag{2.2}$$

Suppose that $\alpha_1, \ldots, \alpha_p$ are equal. Then it follows from (2.1) and (2.2) that $\bar{x} \in D$ is not a solution of (MVVI), which contradicts the fact that $\bar{x} \in sol(\mathrm{MVVI})$.

Suppose that $\alpha_1, \ldots, \alpha_p$ are not equal. Let $\alpha_1 \neq \alpha_2$. From (2.2), we have

$$\langle \xi_1, \bar{x} - z \rangle \geqslant f_1(\bar{x}) - f_1(z) \text{and } \langle \xi_2, \bar{x} - z \rangle \geqslant f_2(\bar{x}) - f_2(z). \tag{2.3}$$

Since f_1 and f_2 are convex, we have

$$\langle \xi_1 - \xi_2^*, z(\alpha_1) - z(\alpha_2) \rangle \geqslant 0 \quad \text{for any} \quad \xi_2^* \in \partial f_1(z(\alpha_2)) \tag{2.4}$$

and

$$\langle \xi_1^* - \xi_2, z(\alpha_1) - z(\alpha_2) \rangle \geqslant 0 \quad \text{for any } \xi_1^* \in \partial f_2(z(\alpha_1)). \tag{2.5}$$

If $\alpha_1 < \alpha_2$, from (2.4), $\langle \xi_1 - \xi_2^*, \bar{x} - z \rangle \leqslant 0$ and hence from (2.3), we have

$$\langle \xi_2^*, \bar{x} - z \rangle \geqslant f_1(\bar{x}) - f_1(z) \quad \text{for any } \xi_2^* \in \partial f_1(z(\alpha_2)).$$

If $\alpha_2 < \alpha_1$, from (2.5), $\langle \xi_1^* - \xi_2, \bar{x} - z \rangle \geqslant 0$ and hence from (2.3), we have

$$\langle \xi_1^*, \bar{x} - z \rangle \geqslant f_2(\bar{x}) - f_2(z) \quad \text{for any } \xi_1^* \in \partial f_2(z(\alpha_1)).$$

Therefore, if $\alpha_1 \neq \alpha_2$, letting $\hat{\alpha}^* = \max\{\alpha_1, \alpha_2\}$, we can find $\bar{\xi}_i \in \partial f_i(z(\hat{\alpha}^*))$, $i = 1, 2$, such that $\langle \bar{\xi}_i, \bar{x} - z \rangle \geqslant f_i(\bar{x}) - f_i(z)$.

By continuing this process, we can find $\hat{\alpha} \in (0, 1)$ and $\bar{\xi}_i \in \partial f_i(z(\hat{\alpha}))$, $i = 1, \ldots, p$, such that

$$\langle \bar{\xi}_i, \bar{x} - z \rangle \geqslant f_i(\bar{x}) - f_i(z). \tag{2.6}$$

From (2.1) and (2.6), $\bar{\xi}_i \in \partial f_i(z(\hat{\alpha}))$, $i = 1, \ldots, p$, and

$$(\langle \bar{\xi}_1, \bar{x} - z \rangle, \ldots, \langle \bar{\xi}_p, \bar{x} - z \rangle) \in \mathbb{R}^p_+ \backslash \{0\}. \tag{2.7}$$

Multiplying both sides of (2.7) by $\hat{\alpha} - 1$, we obtain

$$(\langle \bar{\xi}_1, z(\hat{\alpha}) - \bar{x} \rangle, \ldots, \langle \bar{\xi}_p, z(\hat{\alpha}) - \bar{x} \rangle) \in -\mathbb{R}^p_+ \backslash \{0\},$$

which contradicts the fact that $\bar{x} \in sol(\text{MVVI})$. □

Now we give examples for the relations in Theorem 2.1.

EXAMPLE 2.1. [13] It may not be true that

$$sol(\text{VVI})_2 \subset PrEff(\text{VP}).$$

Let $f(x, y) = (f_1(x, y), f_2(x, y)) = ((1/2)\mu x^2 + (1/2)y^2, (1/2)x^2 + (1/2)y^2)$ and $D := \{(x, y) \in \mathbb{R}^2 \mid (x-2)^2 + (y-2)^2 \leqslant 1\}$, where $\mu = (24\sqrt{7} - 21)/35$.

Then $(\bar{x}, \bar{y}) := (5/4, 2 - (\sqrt{7}/4)) \in sol(\text{VVI})_2 = \{(x, y) \in \mathbb{R}^2 \mid (5/4) \leqslant x \leqslant 2 - (\sqrt{2}/2),\ (x-2)^2 + (y-2)^2 = 1\}$, but $(\bar{x}, \bar{y}) \notin \bigcup_{\lambda \in \text{int}\, \mathbb{R}^p_+} sol(\text{VI})_\lambda = \{(x, y) \in \mathbb{R}^2 \mid (5/4) < x < 2 - (\sqrt{2}/2),\ (x-2)^2 + (y-2)^2 = 1\}$. See Ref. [13] for the calculations of $sol(\text{VVI})_2$ and $\bigcup_{\lambda \in \text{int}\, \mathbb{R}^p_+} sol(\text{VI})_\lambda$. From Theorem 2.1, $\bigcup_{\lambda \in \text{int}\, \mathbb{R}^p_+} sol(\text{VI})_\lambda = PrEff(\text{VP})$. Hence $(\bar{x}, \bar{y}) \notin PrEff(\text{VP})$.

EXAMPLE 2.2. It may not be true that

$$sol(\text{VVI})_3 \subset sol(\text{VVI})_2.$$

Let $f_1(x, y) = \sqrt{x^2 + y^2} + y$, $f_2(x, y) = y$ and $D := \{(x, y) \in \mathbb{R}^2 \mid x \leqslant 0, -\sqrt{-x} \leqslant y \leqslant 0\}$. If $(x, y) = (0, 0)$, $\partial f_1(x, y) = \{(v_1, v_2) \in \mathbb{R}^2 \mid v_1^2 + v_2^2 \leqslant 1\} + \{(0, 1)\} = \{(v_1, v_2) \in \mathbb{R}^2 \mid v_1^2 + (v_2 - 1)^2 \leqslant 1\}$, and if $(x, y) \neq (0, 0)$, $\partial f_1(x, y) = \{(x/\sqrt{x^2 + y^2}, (y/\sqrt{x^2 + y^2}) + 1)\}$.

We can check that $\forall (v_1, v_2) \in \partial f_1(0, 0)$, $\exists (x, y) \in D$ such that

$$(v_1 x + v_2 y, y) \in -\mathbb{R}^2_+ \backslash \{0\},$$

and that $\forall(x, y) \in D$, $\exists(v_1, v_2) \in \partial f_1(0,0)$ such that

$$(v_1 x + v_2 y, y) \notin -\mathbb{R}^2_+ \backslash \{0\}.$$

Hence $(0,0) \in sol(\text{VVI})_3$, but $(0,0) \notin sol(\text{VVI})_2$.

Moreover, $sol(\text{VVI})_2 = \{(x, -\sqrt{-x}\,) \mid x < 0\}$ and $sol(\text{VVI})_3 = \{(x, -\sqrt{-x}\,) \mid x \leqslant 0\}$.

EXAMPLE 2.3. [2] It may not be true that

$$sol(\text{MVVI}) \subset sol(\text{VVI})_3.$$

Let $f_1(x) = x$, $f_2(x) = x^2$ and $D = (-\infty, 0]$.

Since $(x, 0) \in -\mathbb{R}^2_+ \setminus \{0\}$ $\forall x \in (-\infty, 0)$, $0 \notin sol(\text{VVI})_3$. But, since $(x, 2x^2) \notin -\mathbb{R}^2_+ \setminus \{0\}$ $\forall x \in (-\infty, 0]$, $0 \in sol(\text{MVVI})$. Moreover, we can easily check that $sol(\text{VVI})_3 = (-\infty, 0)$ and $sol(\text{MVVI}) = (-\infty, 0]$.

EXAMPLE 2.4. Let $f_1(x) = x$, $f_2(x) = |x|$ and $D = (-\infty, 0]$.

It is clear that $0 \in Eff(\text{VP})$. Since $(f_1(x) - f_1(0))/(f_2(0) - f_2(x)) = 1$ $\forall x \in (-\infty, 0)$, $0 \in PrEff(\text{VP})$. Since there exist $x \in D$ and $\xi \in [-1, 1]$ such that $(x, \xi x) \in -\mathbb{R}^2_+ \setminus \{0\}$, $0 \notin sol(\text{VVI})_1$. Moreover, the above Example 2.1 tells us that the inclusion: $sol(\text{VVI})_1 \subset PrEff(\text{VP})$ may not hold. Hence we can not give any inclusion relation between $sol(\text{VVI})_1$ and $PrEff(\text{VP})$.

THEOREM 2.2. *The following relations hold*

$$\begin{aligned} sol(WVVI)_1 \subset WEff(VP) &= \bigcup_{\lambda \in \mathbb{R}^p_+ \setminus \{0\}} sol(VI)_\lambda = \bigcup_{\lambda \in \mathbb{R}^p_+ \setminus \{0\}} sol(MVI)_\lambda \\ &= sol(WVVI)_2 = sol(WVVI)_3 = sol(WMVVI). \end{aligned}$$

Proof. It can be easily checked that $sol(\text{WVVI})_1 \subset WEff(\text{VP})$. Now we prove that $WEff(\text{VP}) = \bigcup_{\lambda \in \mathbb{R}^p_+ \setminus \{0\}} sol(\text{VI})_\lambda$. By Lemma 1.3, $\bar{x} \in WEff(\text{VP})$ if and only if $\exists \lambda_i \geqslant 0$, $i = 1, \ldots, p$, $(\lambda_1, \ldots, \lambda_p) \neq 0$, such that $\bar{x}$ is a solution of the following scalar optimization problem (SP):

$$(\text{SP}) \quad \begin{array}{ll} \text{Minimize} & \sum_{i=1}^p \lambda_i f_i(x) \\ \text{subject to} & x \in D. \end{array}$$

Thus, we can easily check that $WEff(\text{VP}) = \bigcup_{\lambda \in \mathbb{R}^p_+ \setminus \{0\}} sol(\text{VI})_\lambda = \bigcup_{\lambda \in \mathbb{R}^p_+ \setminus \{0\}} sol(\text{MVI})_\lambda$.

Now we prove that $WEff(\text{VP}) = sol(\text{WMVVI})$.

Let $\bar{x} \notin sol(\text{WMVVI})$.

Then $\exists x^* \in D$ and $\xi_i^* \in \partial f_i(x^*)$, $i = 1, \ldots, p$, such that

$$(\langle \xi_1^*, x^* - \bar{x} \rangle, \ldots, \langle \xi_p^*, x^* - \bar{x} \rangle) \in -\text{int } \mathbb{R}_+^p.$$

Thus $\bar{x} \notin \bigcup_{\lambda \in \mathbb{R}_+^p \setminus \{0\}} sol(\text{MVI})_\lambda$ and hence $\bar{x} \notin WEff(\text{VP})$. Using the method similar to the proof in Theorem 2.1, we can prove that

$$sol(\text{WMVVI}) \subset WEff(\text{VP}).$$

Now we prove that $\bigcup_{\lambda \in \mathbb{R}_+^p \setminus \{0\}} sol(\text{VI})_\lambda = sol(\text{WVVI})_2 = sol(\text{WVVI})_3$.
$\bar{x} \in sol(\text{WVVI})_2$
$\iff \bar{x} \in D$ and $\exists \xi_i \in \partial f_i(\bar{x})$, $i = 1, \ldots, p$, such that

$$\{(\langle \xi_1, x - \bar{x} \rangle, \ldots, \langle \xi_p, x - \bar{x} \rangle) \mid x \in D\} \cap (-\text{int } \mathbb{R}_+^p) = \emptyset$$

$\iff$ (by separation theorem in Ref. [14], Theorem 3.16] $\bar{x} \in D$ and $\exists \xi_i \in \partial f_i(\bar{x})$, $\lambda_i \geqslant 0$, $i = 1, \ldots, p$, $(\lambda_1, \ldots, \lambda_p) \neq 0$ and $r \in \mathbb{R}$ such that

$$\sum_{i=1}^{p} \lambda_i z_i < r \leqslant \sum_{i=1}^{p} \lambda_i \langle \xi_i, x - \bar{x} \rangle \quad \forall x \in D, \quad \forall (z_1, \ldots, z_p) \in -int\, \mathbb{R}_+^p$$

$\iff \bar{x} \in D$ and $\exists \xi_i \in \partial f_i(\bar{x})$, $\lambda_i \geqslant 0$, $i = 1, \ldots, p$, $(\lambda_1, \ldots, \lambda_p) \neq 0$ such that

$$\left\langle \sum_{i=1}^{p} \lambda_i \xi_i, x - \bar{x} \right\rangle \geqslant 0 \quad \forall x \in D$$

$\iff \bar{x} \in \bigcup_{\lambda \in \mathbb{R}_+^p \setminus \{0\}} sol(\text{VI})_\lambda$.

$\bar{x} \in sol(\text{WVVI})_3$
$\iff \bar{x} \in D$ and the system

$$\left\langle \begin{array}{c} \max_{\xi_1 \in \partial f_1(\bar{x})} \langle \xi_1, x - \bar{x} \rangle < 0 \\ \vdots \\ \max_{\xi_p \in \partial f_p(\bar{x})} \langle \xi_p, x - \bar{x} \rangle < 0 \end{array} \right\rangle$$

has no solution $x \in D$
$\iff$ (by Lemma 1.5) $\bar{x} \in D$ and $\exists \lambda_i \geqslant 0$, $i = 1, \ldots, p$, $(\lambda_1, \ldots, \lambda_p) \neq 0$ such that

$$\sum_{i=1}^{p} \lambda_i \max_{\xi_i \in \partial f_i(\bar{x})} \langle \xi_i, x - \bar{x} \rangle \geqslant 0 \quad \forall x \in D$$

$\Longleftrightarrow \bar{x} \in D$ and $\exists \lambda_i \geqslant 0,\ i=1,\ldots,p,\ (\lambda_1,\ldots,\lambda_p) \neq 0$ such that

$$\max_{b \in B} \langle b, x - \bar{x} \rangle \geqslant 0 \quad \forall x \in D,$$

where $B = \{\sum_{i=1}^{p} \lambda_i \xi_i \mid \xi_i \in \partial f_i(\bar{x}),\ i=1,\ldots,p\}$

$\Longleftrightarrow$ (by Lemma 1.4) $\bar{x} \in D$ and $\exists \lambda_i \geqslant 0,\ i=1,\ldots,p,\ (\lambda_1,\ldots,\lambda_p) \neq 0$ and $b \in B$ such that

$$\langle b, x - \bar{x} \rangle \geqslant 0 \quad \forall x \in D$$

$\Longleftrightarrow \bar{x} \in D$ and $\exists \lambda_i \geqslant 0,\ i=1,\ldots,p,\ (\lambda_1,\ldots,\lambda_p) \neq 0,\ \xi_i \in \partial f_i(\bar{x}),\ i = 1,\ldots,p$ such that

$$\left\langle \sum_{i=1}^{p} \lambda_i \xi_i, x - \bar{x} \right\rangle \geqslant 0 \quad \forall x \in D$$

$\Longleftrightarrow \bar{x} \in \bigcup_{\lambda \in \mathbb{R}^p_+ \setminus \{0\}} sol(\mathrm{VI})_\lambda$.

Hence $\bigcup_{\lambda \in \mathbb{R}^p_+ \setminus \{0\}} sol(\mathrm{VI})_\lambda = sol(\mathrm{WVVI})_2 = sol(\mathrm{WVVI})_3$. □

Now we give an example for $(\mathrm{WVVI})_1$.

EXAMPLE 2.5. Let $f_1(x) = x$, $f_2(x) = \begin{cases} x^2, & x < 0 \\ x, & x \geqslant 0 \end{cases}$ and $D = (-\infty, 0]$. Then $sol(\mathrm{WVVI})_1 = (-\infty, 0)$, but $WEff(\mathrm{VP}) = (-\infty, 0]$. Thus the inclusion $WEff(\mathrm{VP}) \subset sol(\mathrm{WVVI})_1$ may not hold.

3. Special cases

Now we consider the special cases for $sol(\mathrm{VVI})_2$ and $sol(\mathrm{VVI})_3$, and $sol(\mathrm{VP})$. For one of the cases, we need the definition for the polyhedral convex function [15]. The convex function g: $\mathbb{R}^n \to \mathbb{R}$ is said to be polyhderal if the epigraph of g is a polyhedral convex subset of $\mathbb{R}^{n+1}$.

PROPOSITION 3.1. *If D is a polyhedral convex set in $\mathbb{R}^n$, then*

$$sol(VVI)_2 = PrEff(VP).$$

Proof. From Theorem 2.1, $PrEff(\mathrm{VP}) \subset sol(\mathrm{VVI})_2$. Let $\bar{x} \in sol(\mathrm{VVI})_2$. Then $\bar{x} \in D$ and $\exists \xi_i \in \partial f_i(\bar{x}),\ i=1,\ldots,p$, such that $\bar{x}$ is an efficient solution of

$$(VP)' \quad \begin{array}{ll} \text{Minimize} & (\langle \xi_1, x \rangle, \ldots, \langle \xi_p, x \rangle) \\ \text{subject to} & x \in D. \end{array}$$

By Lemma 1.2, $\bar{x} \in D$ is a properly efficient solution of (VP)′, and hence by Lemma 1.1, $\exists \lambda_i > 0$, $i = 1, \ldots, p$, such that $\bar{x} \in D$ is a solution of the following scalar optimization problem:

$$\begin{array}{ll} \text{Minimize} & \sum_{i=1}^{p} \lambda_i \langle \xi_i, x \rangle \\ \text{subject to} & x \in D. \end{array}$$

Thus $\bar{x} \in D$ and $\langle \sum_{i=1}^{p} \lambda_i \xi_i, x - \bar{x} \rangle \geqslant 0 \quad \forall x \in D$. So, $\bar{x} \in sol(\text{VI})_\lambda$. Hence it follows from Theorem 2.1 that $\bar{x} \in PrEff(\text{VP})$. □

PROPOSITION 3.2. *If* $D = \{x \in \mathbb{R}^n \mid \langle a_i, x \rangle \leqslant b_i,\ i = 1, \ldots, m\}$, *where* $a_i \in \mathbb{R}^n$ *and* $b_i \in \mathbb{R}$, *and* f_i, $i = 1, \ldots, p$, *are polyhedral and convex, then*

$$sol(VVI)_3 = PrEff(VP).$$

Proof. $\bar{x} \in sol(\text{VVI})_3 \iff \bar{x} \in D$ is an efficient solution of the following convex vector optimization problem

$$\begin{array}{ll} \text{Minimize} & (\max_{\xi_1 \in \partial f_1(\bar{x})} \langle \xi_1, x - \bar{x} \rangle, \ldots, \max_{\xi_p \in \partial f_p(\bar{x})} \langle \xi_p, x - \bar{x} \rangle) \\ \text{subject to} & x \in D. \end{array}$$

$\iff \bar{x} \in D$ and 0 is an efficient solution of the following convex vector optimization problem

$$\begin{array}{ll} \text{Minimize} & (\max_{\xi_1 \in \partial f_1(\bar{x})} \langle \xi_1, x \rangle, \ldots, \max_{\xi_p \in \partial f_p(\bar{x})} \langle \xi_p, x \rangle) \\ \text{subject to} & x \in D - \bar{x}. \end{array}$$

$\iff$ (letting $I(\bar{x}) = \{i \mid \langle a_i, \bar{x} \rangle = b_i\}$)

$\bar{x} \in D$ and 0 is an efficient solution of the following convex vector optimization problem

$$\begin{array}{ll} \text{Minimize} & (\max_{\xi_1 \in \partial f_1(\bar{x})} \langle \xi_1, x \rangle, \ldots, \max_{\xi_p \in \partial f_p(\bar{x})} \langle \xi_p, x \rangle) \\ \text{subject to} & \langle a_i, x \rangle \leqslant 0,\ i \in I(\bar{x}). \end{array}$$

$\iff \bar{x} \in D$ and 0 is a solution of the following scalar optimization problem

$$\begin{array}{ll} \text{Minimize} & \sum_{i=1}^{p} \max_{\xi_i \in \partial f_i(\bar{x})} \langle \xi_i, x \rangle \\ \text{subject to} & \langle a_i, x \rangle \leqslant 0, \quad i \in I(\bar{x}), \\ & \max_{\xi_i \in \partial f_i(\bar{x})} \langle \xi_i, x \rangle \leqslant 0, i = 1, \ldots, p. \end{array}$$

Since f_i, $i = 1, \ldots, p$, are polyhedral and convex, $\partial f_i(\bar{x})$ are polyhedral, covex and compact (Theorem 23.10 in Ref. [15]) and hence $\partial f_i(\bar{x}) = \text{co}\{b_{i1}, \ldots, b_{in(i)}\}$, where $\{b_{i1}, \ldots, b_{in(i)}\}$ is the set of all the extreme points of $\partial f_i(\bar{x})$ and $\text{co}\{b_{i1}, \ldots, b_{in(i)}\}$ is the convex hull of $\{b_{i1}, \ldots, b_{in(i)}\}$.

Notice that $\max_{\xi\in\partial f_i(\bar{x})}\langle\xi_i, x\rangle \leqslant 0 \Longleftrightarrow \langle b_{ij}, x\rangle \leqslant 0,\ j=1,\dots,n(i)$. Thus we have,

$\bar{x}\in sol(\mathrm{VVI})_3$

$\Longleftrightarrow$ $\bar{x}\in D$ and 0 is a solution of the following scalar convex problem

$$\begin{array}{ll} \text{Minimize} & \sum_{i=1}^{p}\max_{\xi_i\in\partial f_i(\bar{x})}\langle\xi_i, x\rangle \\ \text{subject to} & \langle a_i, x\rangle \leqslant 0, i\in I(\bar{x}), \\ & \langle b_{ij}, x\rangle \leqslant 0, i=1,\dots,p,\ j=1,\dots,n(i) \end{array}$$

$\Longleftrightarrow$ $\bar{x}\in D$ and $\exists\lambda_{ij}\geqslant 0, i=1,\dots,p,\ j=1,\dots,n(i), \mu_k\geqslant 0,\ k\in I(\bar{x})$ such that

$$0\in\sum_{i=1}^{p}\partial f_i(\bar{x})+\sum_{i,j}\lambda_{ij}b_{ij}+\sum_{k\in I(\bar{x})}\mu_k a_k$$

$\Longleftrightarrow$ $\bar{x}\in D$ and $\exists\lambda_i\geqslant 0,\ i=1,\dots,p,\ \mu_k\geqslant 0,\ k\in I(\bar{x})$ such that

$$0\in\sum_{i=1}^{p}(1+\lambda_i)\partial f_i(\bar{x})+\sum_{k\in I(\bar{x})}\mu_k a_k$$

$\Longleftrightarrow$ $\bar{x}\in D$ and $\exists\bar{\lambda}_i>0, i=1,\dots,p, \bar{\mu}_k\geqslant 0, k\in I(\bar{x})$ such that

$$0\in\sum_{i=1}^{p}\bar{\lambda}_i\partial f_i(\bar{x})+\sum_{k\in I(\bar{x})}\bar{\mu}_k a_k$$

$\Longleftrightarrow$ (letting $\bar{\mu}_k=0\quad\forall k\notin I(\bar{x})$) $\bar{x}\in D$ and $\exists\bar{\lambda}_i>0, i=1,\dots,p, \bar{\mu}_k\geqslant 0, k\in I(\bar{x})$ such that

$$0\in\sum_{i=1}^{p}\bar{\lambda}_i\partial f_i(\bar{x})+\sum_{k=1}^{m}\bar{\mu}_k a_k \quad\text{and}\quad \bar{\mu}_k(a_k^T\bar{x}-b_k)=0,\ k=1,\cdots,m$$

$\Longleftrightarrow$ $\bar{x}\in D$ and $\exists\bar{\lambda}_i>0, i=1,\dots,p$ such that $\bar{x}$ is a solution of the following scalar optimization problem

$$\begin{array}{ll} \text{Minimize} & \sum_{i=1}^{p}\bar{\lambda}_i f_i(x) \\ \text{subject to} & \langle a_k, x\rangle \leqslant b_k,\ k=1,\dots,m \end{array}$$

$\Longleftrightarrow$ (by Lemma 1.1) $\bar{x}\in PrEff(\mathrm{VP})$.

Hence $sol(\mathrm{VVI})_3 = PrEff(\mathrm{VP})$. □

PROPOSITION 3.3. *If* $sol(VI)_\lambda$ *is nonempty and singleton for any* $\lambda \in \mathbb{R}^p_+\backslash\{0\}$, *then* $Eff(VP) = WEff(VP) = \bigcup_{\lambda \in \mathbb{R}^p_+\backslash\{0\}} (VI)_\lambda$.

Proof. We know that $Eff(\mathrm{VP}) \subset WEff(\mathrm{VP})$. Let $\bar{x} \in WEff(\mathrm{VP})$. Then by Theorem 2.2, there exists $\lambda \in \mathbb{R}^p_+\backslash\{0\}$ such that $\bar{x} \in sol(\mathrm{VI})_\lambda$. Thus, $\exists \xi_i \in \partial f_i(\bar{x})$, $i = 1, \ldots, p$, such that

$$\left\langle \sum_{i=1}^p \lambda_i \xi_i, x - \bar{x} \right\rangle \geqslant 0 \quad \forall x \in D.$$

Suppose that $x^* \in D$ and $(f_1(x^*), \ldots, f_p(x^*)) - (f_1(\bar{x}), \ldots, f_p(\bar{x})) \in -\mathbb{R}^p_+$. Then $\sum_{i=1}^p \lambda_i f_i(x^*) \leqslant \sum_{i=1}^p \lambda_i f_i(\bar{x})$ for any $x \in D$. Since f_i is convex, we have

$$\begin{aligned}\left\langle \sum_{i=1}^p \lambda_i \xi_i, x^* - \bar{x} \right\rangle &\leqslant \sum_{i=1}^p \lambda_i f_i(x^*) - \sum_{i=1}^p \lambda_i f_i(\bar{x}) \\ &\leqslant 0.\end{aligned}$$

Hence, for any $x \in D$,

$$\begin{aligned}\left\langle \sum_{i=1}^p \lambda_i \xi_i, x - x^* \right\rangle &= \left\langle \sum_{i=1}^p \lambda_i \xi_i, x - \bar{x} \right\rangle + \left\langle \sum_{i=1}^p \lambda_i \xi_i, \bar{x} - x^* \right\rangle \\ &\geqslant \left\langle \sum_{i=1}^p \lambda_i \xi_i, x - \bar{x} \right\rangle \\ &\geqslant 0.\end{aligned}$$

Thus $x^* \in sol(\mathrm{VI})_\lambda$. Since $sol(\mathrm{VI})_\lambda$ is singleton, $x^* = \bar{x}$ and hence $f_i(x^*) = f_i(\bar{x})$, $i = 1, \ldots, p$. Thus $\bar{x} \in Eff(\mathrm{VP})$. Consequently, $Eff(\mathrm{VP}) = WEff(\mathrm{VP})$. By Theorem 2.2, $Eff(\mathrm{VP}) = WEff(\mathrm{VP}) = \bigcup_{\lambda \in \mathbb{R}^p_+\backslash\{0\}} sol(\mathrm{VI})_\lambda$. □

REMARK 3.1. If $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, p$, are continuously differentiable and strongly convex (see Ref. [16] for the definition of the strong convexity) and $\nabla f_i(\cdot)$, $i = 1, \ldots, p$, are Lipschitz on D, then $sol(\mathrm{VI})_\lambda$ is nonempty and singleton for any $\lambda \in \mathbb{R}^p_+\backslash\{0\}$.

The following example comes from Ref. [17].

EXAMPLE 3.1. The assumption of Proposition 3.3 is essential. Let $f_1(x, y) = (1/2)x^2$, $f_2(x, y) = (1/2)y^2$ and $D = \{(x, y) \in \mathbb{R}^2 |\ 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1\}$.

Then $sol(\mathrm{VI})_\lambda$ is nonempty for any $\lambda \in \mathbb{R}^p_+ \setminus \{0\}$. Moreover, $sol(\mathrm{VI})_{(1,0)} = \{(0, y) \in \mathbb{R}^2 \mid 0 \leqslant y \leqslant 1\}$, and hence $sol(\mathrm{VI})_{(1,0)}$ is not a singleton. However, $Eff(\mathrm{VP}) = \{(0, 0)\}$ and $WEff(\mathrm{VP}) = \bigcup_{\lambda \in \mathbb{R}^p_+ \setminus \{0\}} sol(\mathrm{VI})_\lambda = \{(x, 0) \in \mathbb{R}^2 \mid 0 \leqslant x \leqslant 1\} \cup \{(0, y) \in \mathbb{R}^2 \mid 0 \leqslant y \leqslant 1\}$.

DEFINITION 3.1. A subset $M \subset \mathbb{R}^n$ is said to be a strictly convex body if int $M \neq \emptyset$, and for any $x, x' \in M$, $x \neq x'$,

$$\{\lambda x + (1-\lambda)x' | \lambda \in (0, 1)\} \subset \text{ int } M.$$

Following the approach of the proof in Theorem 2 in Ref. [18], we can obtain the following proposition

PROPOSITION 3.4. *Suppose that*

(*i*) $\bar{x} \in sol(WVVI)_1$,
(*ii*) *there exist* $\xi_i \in \partial f_i(\bar{x})$, $i = 1, \ldots, p$, *such that the linear operator* $v \mapsto (\langle \xi_1, v\rangle, \ldots, \langle \xi_p, v\rangle)$ *is surjective, and*
(*iii*) *the constraint set* D *is a strictly convex body in* $\mathbb{R}^n$.

Then $\bar{x} \in sol(VVI)_3$ *and hence* $\bar{x} \in Eff(VP)$.

Proof. Let $\xi_i \in \partial f_i(\bar{x})$, $i = 1, \ldots, p$ be such that ξ_i is in assumption (ii) and $\lambda \in (0, 1)$ and $\Lambda(v) = (\langle \xi_1, v\rangle, \ldots, \langle \xi_p, v\rangle)$ for any $v \in \mathbb{R}^n$. Then $\Lambda: \mathbb{R}^n \to \mathbb{R}^p$ is a continuous and surjective linear operator.

Suppose to the contrary that $\bar{x} \notin sol(\mathrm{VVI})_3$. Then we can choose $z \in D$ such that

$$\Lambda(z - \bar{x}) \in -\mathbb{R}^p_+ \setminus \{0\}. \tag{3.1}$$

Moreover $z_\lambda := \lambda z + (1-\lambda)\bar{x} \in \text{int } D$ since D is a strictly convex body. Thus there exists $\epsilon > 0$ such that

$$B(z_\lambda, \epsilon) \subset D,$$

where $B(z_\lambda, \epsilon)$ is the closed ball centered at z_λ with radius ϵ. Let $y_\lambda = \Lambda(z_\lambda - \bar{x})$. Then $y_\lambda = \lambda\Lambda(z - \bar{x})$, and hence it follows from (3.1) that

$$y_\lambda \in -\mathbb{R}^p_+ \setminus \{0\}. \tag{3.2}$$

By open mapping theorem, $\Lambda(B(z_\lambda, \epsilon) - \bar{x})$ is a neighborhood of y_λ. Thus there exists $\rho > 0$ such that

$$B(y_\lambda, \rho) \subset \Lambda(B(z_\lambda, \epsilon) - \bar{x}). \tag{3.3}$$

From (3.2), $y_\lambda \in B(y_\lambda, \rho) \cap (-\mathbb{R}^p_+)$. So, by Corollary 6.3.2 in Ref. [15],

$$B(y_\lambda, \rho) \cap (-\text{int } \mathbb{R}^p_+) \neq \emptyset.$$

Let $y^* \in B(y_\lambda, \rho) \cap (-\text{int } \mathbb{R}^p_+)$. Then from (3.3), there exists $x^* \in D$ such that $\Lambda(x^* - \bar{x}) \in -\text{int } \mathbb{R}^p_+$. This means that $\bar{x} \notin sol(\text{WVVI})_1$. This contradicts the assumption (i). Consequently, $\bar{x} \in sol(\text{VVI})_3$. It follows from Theorem 2.1 that $\bar{x} \in Eff(\text{VP})$. □

EXAMPLE 3.2. Let $f_1(x, y) = x$, $f_2(x, y) = \sqrt{x^2 + (y-1)^2} - y$ and $D = \{(x, y) \in \mathbb{R}^2 \mid (x-1)^2 + (y-1)^2 \leqslant 1\}$. Then $\partial f_1(0, 1) = \{(1, 0)\}$ and $\partial f_2(0, 1) = \{(x, y) \in \mathbb{R}^2 \mid x^2 + (y+1)^2 \leqslant 1\}$. We can easily check that $(0, 1) \in sol(\text{WVVI})_1$, and that assumptions (ii) and (iii) are satisfied. Hence it follows from Proposition 3.4 that $(0, 1) \in sol(\text{VVI})_3$ and $(0, 1) \in Eff(\text{VP})$.

PROPOSITION 3.5. If there exists $i \in \{1, \ldots, p\}$ such that the function f_i is strictly convex and $\bar{x} \in sol(\text{VI})_\lambda$, where $\lambda = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^p_+$ and 1 is the ith component of λ, then $\bar{x} \in Eff(\text{VP})$.

Proof. Since $\bar{x} \in sol(\text{VI})_\lambda$, there exists $\xi_i \in \partial f_i(\bar{x})$ such that

$$\langle \xi_i, x - \bar{x} \rangle \geqslant 0 \quad \forall x \in D$$

and hence by the strict convexity of f_i,

$$f_i(x) > f_i(\bar{x}) \quad \forall x \in D.$$

Hence $\bar{x} \in Eff(\text{VP})$. □

REMARK 3.2. Let us consider Example 2.2 again. Since $(0, 0) \in \partial f_1(0, 0)$, it is obvious that $(0, 0) \in sol(\text{VI})_{(1,0)}$. Since f_1 is strictly convex, it follows from Proposition 3.5 that $(0, 0) \in Eff(\text{VP})$.

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